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Trace class multipliers and spectral variation of normal matrices

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Abstract

In this article we show how to estimate the trace multiplier norm of a rank 2 matrix. As an application, an alternative proof of a theorem of Holbrook et al. (Maximal spectral distance, Linear Algebra Appl., 249 (1996) 197–205) on the maximal spectral distance between two normal matrices with prescribed eigenvalues is given. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $M(p,q)$ denote the space of $p \times q$ complex matrices. This space can be endowed with a wide variety of norms. Among the most interesting are the operator norm

$$|||A||| = \max_{j=1}^{\min(p,q)} \sigma_j(A)$$

and its dual, the trace norm

$$\|A\|_{\mathrm{TR}} = \sum_{j=1}^{\min(p,q)} \sigma_j(A).$$

In the above definitions we have denoted by $\sigma_j(A)$ the jth singular value of the $p \times q$ complex matrix A. For details the reader should consult ([1], Section 3.5).

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A further norm, which plays an important role in the theory of spectral variation is the trace class multiplier norm

$$\|A\|_{\mathrm{TM}} = \sup_{\|B\|_{\mathrm{TR}} \leqslant 1} \|A \circ B\|_{\mathrm{TR}} \tag{1.1}$$

or equivalently one can obtain

$$\|A\|_{\mathrm{TM}} = \sup_{|||B||| \leqslant 1} |||A \circ B||| \tag{1.2}$$

by duality. Here the notation $A \circ B$ denotes the Hadamard product of A and B.

The primary objective of this article is to use this equivalence to give an alternative proof of a theorem of Holbrook et al. ([2], Theorem 2.4) on the maximal spectral distance between two normal matrices with prescribed eigenvalues. This new proof replaces the beautiful argument of Omladič and Šemrl [3], Theorem 3.1 by one which relies only on elementary calculus. We also obtain some generalizations.

2. Trace class multipliers

An important estimate arises in the case that $a_{ij} = \xi_i \eta_j$ and we have the estimate

$$|a_{ij}| \leqslant 1, \qquad i = 1, \dots, p, \quad j = 1, \dots, q. \tag{2.1}$$

Then, it is easy to see that we can assume without loss of generality that the estimates

$$|\xi_i| \leqslant 1 \qquad i = 1, \dots, p, \qquad\qquad |\eta_j| \leqslant 1 \qquad j = 1, \dots, q \tag{2.2}$$

hold and it follows that $\|A\|_{\mathrm{TM}} \leqslant 1$. We can summarize the above by the following lemma.

Lemma 2.1. *Let A be a rank one matrix. Then*

$$\|A\|_{\mathrm{TM}} \leqslant \max_{i,j} |a_{i,j}|.$$

The lemma is false if the rank assumption is dropped.

Consider now the closed convex hull of the set of matrices $a_{ij} = \xi_i \eta_j$ where Eq. (2.2) holds. This is the unit ball for the norm $\| \, \|_{\mathrm{V}}$ introduced in Grothendieck [4] and popularized by Varopoulos [5]. It is a consequence of Lemma 2 that

$$\|A\|_{\mathrm{TM}} \leqslant \|A\|_{\mathrm{V}}.$$

A quite remarkable fact ([6], Theorem 6.4) is that there exists a universal constant $C > 1$ (independent of p and q) such that

$$\|A\|_{\mathrm{V}} \leqslant C\|A\|_{\mathrm{TM}}.$$

We present in Lemma 2.2 our basic strategy for estimating trace class multiplier norms.

Lemma 2.2. *Let A be a $p \times q$ complex matrix. Then*

$$\|A\|_{\mathrm{TM}}^2 \leqslant \|A\|_\infty^2 + \|A \wedge A\|_{\mathrm{TM}}.$$

Here, we have denoted

$$\|A\|_\infty = \max_{i=1}^{p} \max_{k=1}^{q} |a_{ik}|,$$

the uniform norm of the matrix A. The notation $A \wedge A$ denotes the exterior product of A with itself, or alternatively the matrix of 2×2 minors of A.

Proof. Let $b_{ik} = \xi_i \eta_k$, where ξ and η are unit vectors. Let $C = B \circ A$ and $H = C^\star C$. Let $r = \min(p, q)$ and we denote by $\sigma_1, \ldots, \sigma_r$ the singular values of C. We have

$$\begin{aligned} \|C\|_{\mathrm{TR}}^2 &= \left(\sum_{i=1}^{r} \sigma_i\right)^2 \\ &= \sum_{i=1}^{r} \sigma_i^2 + 2\sum_{i<j} \sigma_i \sigma_j \\ &= \mathrm{tr(H)} + 2\|\mathrm{C} \wedge \mathrm{C}\|_{\mathrm{TR}}. \end{aligned} \tag{2.3}$$

Now, for the first term

$$\mathrm{tr}(H) = \sum_{i=1}^{p} \sum_{k=1}^{q} |a_{ik}|^2 |\xi_i|^2 |\eta_k|^2 \leqslant \|A\|_\infty^2.$$

And for the second, we find

$$(C \wedge C)_{(ij)(k\ell)} = c_{ik}c_{j\ell} - c_{i\ell}c_{jk} = \xi_i \xi_j (a_{ik}a_{j\ell} - a_{i\ell}a_{jk}) \eta_k \eta_\ell.$$

So using the definition of the trace multiplier norm we find

$$\|C \wedge C\|_{\mathrm{TR}} \leqslant \left\{\sum_{i<j} |\xi_i \xi_j|^2\right\}^{1/2} \|A \wedge A\|_{\mathrm{TM}} \left\{\sum_{k<\ell} |\eta_k \eta_\ell|^2\right\}^{1/2}.$$

However

$$2\sum_{i<j} |\xi_i \xi_j|^2 \leqslant \sum_{i=1}^{p} |\xi_i|^2 \sum_{j=1}^{p} |\xi_j|^2 \leqslant 1$$

with a corresponding inequality for the η's. We get

$$\|C\|_{\mathrm{TR}}^2 \leqslant \|A\|_\infty^2 + \|A \wedge A\|_{\mathrm{TM}}.$$

Finally, taking the supremum over all ξ and η we have the conclusion of the lemma. □

In case that A has rank at most 2 we can be much more precise.

Lemma 2.3. *Let A be a $p \times q$ complex matrix with* $\operatorname{rank}(A) \leqslant 2$. *Then*

$$\|A\|_{\mathrm{TM}}^2 = \sup_{t,s}\left(\sum_{j,\ell} t_j s_\ell |a_{j\ell}|^2 + 2\sqrt{\sum_{\substack{j<k\\ \ell<m}} t_j t_k s_\ell s_m |a_{j\ell}a_{km} - a_{jm}a_{k\ell}|^2}\right), \tag{2.4}$$

where the supremum is taken over all nonnegative p-tuples (t_j) summing to 1 *and all nonnegative q-tuples (s_ℓ) summing to* 1.

Proof. We proceed as in Lemma 2.2 but with the important difference that $C \wedge C$ now has rank 1. Therefore we can replace the trace norm $\|C \wedge C\|_{\mathrm{TR}}$ with the Frobenius norm $\|C \wedge C\|_{\mathrm{HS}}$ in Eq. (2.3) to obtain the expression

$$\sum_{j,\ell} |\xi_j|^2 |\eta_\ell|^2 |a_{j\ell}|^2 + 2\sqrt{\sum_{\substack{j<k\\ \ell<m}} |\xi_j|^2 |\xi_k|^2 |\eta_\ell|^2 |\eta_m|^2 |a_{j\ell}a_{km} - a_{jm}a_{k\ell}|^2}$$

for $\|C\|_{\mathrm{TR}}^2$. We substitute $t_j = |\xi_j|^2$ and $s_\ell = |\eta_\ell|^2$ and take the supremum over ξ and η to obtain the conclusion (2.4). □

3. Spectral distance and related estimates

In the maximal spectral distance problem, complex numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are given and a sharp upper bound

$$F_n(a_1, \ldots, a_n; b_1, \ldots, b_n)$$

is sought for $|||A - B|||$ subject to the constraint that A and B are $n \times n$ normal matrices with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. This question has been completely solved by Holbrook et al. [3], Theorem 2.4 who prove in case $n \geqslant 2$ that

$$F_n(a_1, \ldots, a_n; b_1, \ldots, b_n) = \sup_{j,k,\ell,m} F_2(a_j, a_k; b_\ell, b_m) \tag{3.1}$$

effectively reducing the problem to the case $n = 2$. They also give an explicit geometrical description of F_2. In Theorem 3.1 we give an alternative approach to their result. We are grateful to the referee for pointing out that back in

1970, using maximal numerical range, Stampfli [7], Theorem 8 obtained the equality

$$\sup_{|||X|||\leqslant 1} |||AX - XB||| = \inf_{\lambda\in\mathbb{C}}\{|||A - \lambda I||| + |||\lambda I - B|||\}$$

for A, B and X operators on a complex Hilbert space. The maximal spectral distance problem follows easily from his result.

Theorem 3.1. *The equality* (3.1) *holds in case* $n \geqslant 2$.

Proof. In estimating $|||A - B|||$, we can always assume that

$$A = U^\star \mathrm{diag}(a_1, \ldots, a_n)U \quad \text{and} \quad B = \mathrm{diag}(b_1, \ldots, b_n),$$

where U is a unitary matrix. Thus

$$\begin{aligned} |||A - B||| &= |||U^\star\mathrm{diag}(a_1, \ldots, a_n)U - \mathrm{diag}(b_1, \ldots, b_n)||| \\ &= |||\mathrm{diag}(a_1, \ldots, a_n)U - U\mathrm{diag}(b_1, \ldots, b_n)||| \\ &= |||D \circ U|||, \end{aligned}$$

where $d_{j\ell} = a_j - b_\ell$. We are therefore required to find

$$\sup_{U\in U(n)} |||D \circ U|||.$$

But it is well known (and an immediate consequence of the Singular Value Decomposition) that the convex hull of the set $U(n)$ of all unitary $n \times n$ matrices is the unit ball for the operator norm. Thus we have

$$\sup_{U\in U(n)} |||D \circ U||| = \sup_{|||U|||\leqslant 1} |||D \circ U||| = \|D\|_{\mathrm{TM}}$$

by Eq. (1.2). Since D is manifestly a matrix of rank 2 at most, we can use Lemma 2.3 to obtain

$$\|D\|^2_{\mathrm{TM}} = \sup_{t,s} f(t, s),$$

where

$$\begin{aligned} f(t,s) &= \sum_{j,\ell} t_j s_\ell |a_j - b_\ell|^2 + 2\sqrt{\sum_{\substack{j<k \\ \ell<m}} t_j t_k s_\ell s_m |a_j - a_k|^2 |b_\ell - b_m|^2} \\ &= \sum_{j,\ell} t_j s_\ell |a_j - b_\ell|^2 + 2g(t)h(s), \end{aligned} \tag{3.2}$$

where

$$g(t) = \sqrt{\sum_{j<k} t_j t_k |a_j - a_k|^2} \quad \text{and}$$

$$h(s) = \sqrt{\sum_{\ell<m} s_\ell s_m |b_\ell - b_m|^2}.$$

We now use the method of Lagrange multipliers to track down possible maximum points of f. Since f is homogenous of degree 1 in both t and s we have

$$\sum_j t_j \frac{\partial f}{\partial t_j} = f \quad \text{and} \quad \sum_\ell s_\ell \frac{\partial f}{\partial s_\ell} = f. \tag{3.3}$$

At a local maximum point, for each j either $t_j = 0$ or $\partial f/\partial t_j = \lambda$ the Lagrange multiplier. It follows from Eq. (3.3) that for each j, either $t_j = 0$ or $\partial f/\partial t_j = f$ and, similarly for each ℓ either $s_\ell = 0$ or $\partial f/\partial s_\ell = f$.

To simplify expressions in forthcoming calculations we denote

$$\alpha = \sum_j t_j a_j, \quad \beta = \sum_\ell s_\ell b_\ell, \quad \tau = \sum_j t_j |a_j|^2, \quad \sigma = \sum_\ell s_l |b_\ell|^2.$$

Thus, we get

$$\begin{aligned} \frac{\partial f}{\partial t_j} &= \sum_\ell s_l |a_j - b_\ell|^2 + g^{-1} h \sum_k t_k |a_j - a_k|^2 \\ &= |a_j|^2 - 2\mathscr{R} a_j \bar{\beta} + \sigma + g^{-1} h(|a_j|^2 - 2\mathscr{R} a_j \bar{\alpha} + \tau). \end{aligned}$$

Thus, for each j either $t_j = 0$ or

$$(g+h)|a_j|^2 - 2\mathscr{R} a_j (g\bar{\beta} + h\bar{\alpha}) + (g\sigma + h\tau - gf) = 0,$$

and similarly for each ℓ, either $s_\ell = 0$ or

$$(g+h)|b_\ell|^2 - 2\mathscr{R} b_\ell (g\bar{\beta} + h\bar{\alpha}) + (g\sigma + h\tau - hf) = 0.$$

We observe that, as if by a miracle, the coefficients of $|a_j|^2$ and $|b_\ell|^2$ are identical, as are those of a_j and b_ℓ. Only the constant coefficients are different. It then follows that there are concentric circles C_A and C_B in the complex plane such that for all j, either

$$t_j = 0 \quad \text{or} \quad a_j \in C_A \tag{3.4}$$

and for all ℓ, either

$$s_\ell = 0 \quad \text{or} \quad b_\ell \in C_B. \tag{3.5}$$

The proof is now completed as in [2]. They work with *generic* a_j and b_ℓ with the additional property that no two b_ℓ's are equidistant from the circumcentre of 3 a_j's and no two a_j's are equidistant from the circumcentre of 3 b_ℓ's. Conditions (3.4) and (3.5) then force one of the three following scenarios.

- $t_j = 0$ for all but one value of j.
- $s_\ell = 0$ for all but one value of ℓ.
- $t_j = 0$ for all but two values of j and $s_\ell = 0$ for all but two values of ℓ.

The third case is exactly what is desired. In the first case $g = 0$ and in the second $h = 0$, and both of these force the desired conclusion by examining Eq. (3.2). Thus the result is proved in the generic case. The general case is established by approximating it with the generic one. □

4. The general rank 2 case

Theorem 4.1. *Let A be a $p \times q$ complex matrix with* $\operatorname{rank}(A) \leqslant 2$. *Then we have*

$$\|A\|_{\mathrm{TM}} = \max_{X,Y} \|A(X,Y)\|_{\mathrm{TM}},$$

where $A(X,Y)$ denotes the submatrix of A corresponding to the row set X and the column set Y and the maximum is taken over all such sets with either $|X| \leqslant 3$ and $|Y| \leqslant 2$ or $|X| \leqslant 2$ and $|Y| \leqslant 3$.

Proof. We use Lemma 2.3 and follow the proof of Theorem 3.1. Let us write

$$a_{j\ell} = a_j c_\ell - b_j d_\ell$$

so that

$$a_{j\ell} a_{km} - a_{jm} a_{k\ell} = (a_j b_k - a_k b_j)(c_\ell d_m - c_m d_\ell).$$

We use the notations

$$g(t) = \sqrt{\sum_{j<k} t_j t_k |a_j b_k - a_k b_j|^2}, \qquad h(s) = \sqrt{\sum_{\ell<m} s_\ell s_m |c_\ell d_m - c_m d_\ell|^2},$$

$$f = \sum_{j,\ell} t_j s_\ell |a_j c_\ell - b_j d_\ell|^2 + 2g(t)h(s)$$

together with

$$\alpha = \sum_j t_j |a_j|^2, \qquad \beta = \sum_j t_j a_j \bar{b}_j, \qquad \gamma = \sum_j t_j |b_j|^2,$$
$$\lambda = \sum_\ell s_\ell |c_\ell|^2, \qquad \mu = \sum_\ell s_\ell c_\ell \bar{d}_\ell, \qquad \nu = \sum_\ell s_\ell |d_\ell|^2.$$

We then have

$$\begin{aligned}\frac{\partial f}{\partial t_j} &= \sum_\ell s_l |a_j c_\ell - b_j d_\ell|^2 + g^{-1} h \sum_k t_k |a_j b_k - a_k b_j|^2 \\ &= \lambda |a_j|^2 - 2\mathscr{R}\mu a_j \bar{b}_j + \nu |b_j|^2 + g^{-1} h \Big(\gamma |a_j|^2 - 2\mathscr{R}\bar{\beta} a_j \bar{b}_j + \alpha |b_j|^2\Big),\end{aligned}$$

so that at a critical point, for each j, either $t_j = 0$ or

$$(g\lambda + h\gamma)|a_j|^2 - 2\mathscr{R}(g\mu + h\bar{\beta})a_j\bar{b}_j + (g\nu + h\alpha)|b_j|^2 = gf$$

and similarly, for each ℓ, either $s_\ell = 0$ or

$$(g\nu + h\alpha)|c_\ell|^2 - 2\mathscr{R}(g\mu + h\bar{\beta})\bar{c}_\ell d_\ell + (g\lambda + h\gamma)|d_\ell|^2 = hf.$$

It follows that the submatrix of the $(p+q) \times 6$ matrix

$$Z = \begin{pmatrix} |a_1|^2 & |b_1|^2 & \mathscr{R}a_1\bar{b}_1 & \Im a_1\bar{b}_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ |a_p|^2 & |b_p|^2 & \mathscr{R}a_p\bar{b}_p & \Im a_p\bar{b}_p & 1 & 0 \\ |d_1|^2 & |c_1|^2 & \mathscr{R}\bar{c}_1 d_1 & \Im\bar{c}_1 d_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ |d_q|^2 & |c_q|^2 & \mathscr{R}\bar{c}_q d_q & \Im\bar{c}_q d_q & 0 & 1 \end{pmatrix}$$

corresponding to those rows for which $t_j > 0$ or $s_\ell > 0$ has rank at most 5.

To proceed further, we need to consider points

$$\theta = (a_1, \ldots, a_p, b_1, \ldots, b_p, c_1, \ldots, c_q, d_1, \ldots, d_q)$$

of $\mathbb{C}^{2(p+q)}$ and define what it means for them to be *generic*. We will say that a pair (X, Y) is *involving* if X and Y are *nonempty* subsets of $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$, respectively such that $|X| + |Y| = 6$. To each involving pair (X, Y) we have a corresponding *involving* 6×6 *submatrix* $Z_{X,Y}$ of the matrix Z. We will say that θ is generic if and only if every involving 6×6 submatrix of Z is nonsingular. The key point that needs to be established is that the set of generic elements is dense in the complete space $\mathbb{C}^{2(p+q)}$. This is established by invoking the Baire Category Theorem. The set of nongeneric elements of $\mathbb{C}^{2(p+q)}$ is a finite union of closed sets

$$\Gamma_{X,Y} = \{\theta; \det(Z_{X,Y}) = 0\}, \quad (X, Y) \text{ involving}$$

and it remains only to show that each such set has empty interior. But $\det(Z_{X,Y})$ is a polynomial function of θ (now viewed on $\mathbb{R}^{4(p+q)}$) and it follows that if $\Gamma_{X,Y}$ has nonempty interior, then $\det(Z_{X,Y})$ vanishes identically. But since (X, Y) is involving, this is easily seen to be impossible.

We now assume that θ is generic. Let

$$\tilde{X} = \{j; t_j > 0\} \quad \text{and} \quad \tilde{Y} = \{\ell; s_\ell > 0\}.$$

Then $|\tilde{X}| + |\tilde{Y}| \geqslant 6$ is impossible, for we would then be able to select an involving pair (X, Y) with $X \subseteq \tilde{X}$ and $Y \subseteq \tilde{Y}$. In the case $|\tilde{X}| = 1$, we take $\tilde{X} = \{j\}$ for some fixed j leading to

$$f = \sum_{\ell} s_\ell |a_j c_\ell + b_j d_\ell|^2.$$

It is clear that f will attain its maximum when $s_\ell = 1$ for some fixed value of ℓ. Thus the norm is attained on a 1×1 block. The case $|\tilde{Y}| = 1$ leads to the same conclusion. In the remaining cases either $|\tilde{X}| \leqslant 2$ and $|\tilde{Y}| \leqslant 3$ or $|\tilde{X}| \leqslant 3$ and $|\tilde{Y}| \leqslant 2$. This establishes the result in the generic case.

The general case follows by approximation with the generic one and since the norms involved are continuous functions of θ. $\square$

Corollary 4.1. *Given complex numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$, let*

$$G_n(a_1, \dots, a_n; b_1, \dots, b_n) = \sup |||I - AB|||,$$

where the supremum is taken over all $n \times n$ normal matrices A and B with eigenvalues $a_1, \dots, a_n$, and $b_1, \dots, b_n$ respectively. Then

$$\begin{aligned} &G_n(a_1, \dots, a_n; b_1, \dots, b_n) \\ &\quad = \sup_{\substack{\ell<m \\ i<j<k}} \max(H(a_\ell, a_m, b_i, b_j, b_k), H(b_\ell, b_m, a_i, a_j, a_k)), \end{aligned}$$

where $H(c_1, c_2, d_1, d_2, d_3)$ denotes the trace multiplier norm of the 2×3 matrix $(1 + c_j d_k)_{j,k}$.

The example $n = 3$, $a_1 = \frac{3}{2}$, $a_2 = i$, $a_3 = -i$, $b_1 = -1$, $b_2 = \frac{1}{2} + \frac{1}{4} i$ and $b_3 = 2$ of the corollary is one in which the maximum over 2×2 blocks does not yield $G_3(a_1, a_2, a_3, b_1, b_2, b_3)$.

5. Computing trace multiplier norms

In [2], Theorem 2.1, the authors use a clever argument to obtain an explicit formula for the quantity $F_2(a_1, a_2, b_1, b_2)$ appearing in Eq. (3.1). In our more general situation, matters are more complicated and we cannot expect so neat a result. The question arises as to how one can compute numerically the trace multiplier norm of a 2×3 matrix.

Before we begin, it is worth observing that the general 2×2 case does present some simplification arising from the fact that there is only one term under the root sign in Eq. (2.4).

Proposition 5.1. *Let A be a 2×2 matrix. Then we have*

$$\|A\|_{\mathrm{TM}}^2 = \sup_{\xi,\eta} \left(2|\det(A)|\xi_1 \xi_2 \eta_1 \eta_2 + \sum_{j=1}^{2} \sum_{\ell=1}^{2} |a_{j\ell}|^2 \xi_j^2 \eta_\ell^2 \right),$$

where the supremum is taken over all real vectors ξ and η on the unit circle in $\mathbb{R}^2$.

In the 2×3 case the corresponding expression for $\|A\|^2_{\mathrm{TM}}$ is

$$\sup_{\xi,\eta}\left(\sum_{j=1}^{2}\sum_{\ell=1}^{3}|a_{j\ell}|^2\xi_j^2\eta_\ell^2 + 2\xi_1\xi_2\sqrt{\sum_{\ell<m}|a_{1\ell}a_{2m} - a_{1m}a_{2\ell}|^2\eta_\ell^2\eta_m^2}\right),$$

where the supremum is taken over all ξ in the unit circle in $\mathbb{R}^2$ and all η in the unit sphere in $\mathbb{R}^3$. Since the quantity in parentheses is a quadratic form in ξ we find that

$$\|A\|^2_{\mathrm{TM}} = \sup_{\eta} |||C(\eta)|||,$$

where $C(\eta)$ is the 2×2 matrix with entries

$$c_{jj} = \sum_{\ell=1}^{3}|a_{j\ell}|^2\eta_\ell^2, \quad j = 1,2$$

and

$$c_{12} = c_{21} = \sqrt{\sum_{\ell<m}|a_{1\ell}a_{2m} - a_{1m}a_{2\ell}|^2\eta_\ell^2\eta_m^2}.$$

However

$$|||C||| = \frac{1}{2}\left(c_{11} + c_{22} + \sqrt{(c_{11} - c_{22})^2 + 4c_{12}c_{21}}\right)$$

so that we can express

$$\|A\|^2_{\mathrm{TM}} = \sup_{s}\left(\sum_{j=1}^{3}\alpha_j s_j + \sqrt{\sum_{j=1}^{3}\sum_{k=1}^{3}\beta_{jk}s_j s_k}\right)$$

the supremum being taken over nonnegative (s_1, s_2, s_3) summing to 1 and where the vector (α_j) and the symmetric matrix (β_{jk}) are suitably chosen. Writing now

$$f(s) = \sum_{j=1}^{3}\alpha_j s_j + g(s), \qquad g(s) = \sqrt{\sum_{j=1}^{3}\sum_{k=1}^{3}\beta_{jk}s_j s_k},$$

we obtain at a critical point that for each $j = 1, 2, 3$ either $s_j = 0$ or

$$f = \alpha + g^{-1}\sum_{k=1}^{3}\beta_{jk}s_k.$$

Three of the corresponding seven cases are easy to handle. To show how to handle the remaining four, we illustrate the case $s_1, s_2, s_3 > 0$. We obtain the linear system of equations

$$\sum_k (\beta_{2k} - \beta_{1k}) s_k = (\alpha_1 - \alpha_2) g,$$

$$\sum_k (\beta_{3k} - \beta_{1k}) s_k = (\alpha_1 - \alpha_3) g,$$

$$\sum_k s_k = 1,$$

and solve it in the form $s_k = \sigma_k + \tau_k g$. Here the σ_k and τ_k are known explicitly, but g has yet to be determined. This is done by solving the quadratic equation

$$g^2 = \sum_{jk} \beta_{jk} (\sigma_j + \tau_j g)(\sigma_k + \tau_k g).$$

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